ON CLASS SUMS IN p-ADIC GROUP RINGS

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1. Introduction. In this note we prove that an isomorphism of p-adic group rings of finite p-groups maps class sums onto class sums. For integral group rings this is a well known theorem of Glauberman (see [3; 7]). As an application, we show that any automorphism of the p-adic group ring of a finite p-group of nilpotency class 2 is composed of a group automorphism and a conjugation by a suitable element of the p-adic group algebra. This was proved for integral group rings of finite nilpotent groups of class 2 in [5]. In general this question remains open. We also indicate an extension of a theorem of Passman and Whitcomb. The following notation is used.

\[ G \text{ denotes a finite } p\text{-group.} \]
\[ Z \text{ denotes the ring of (rational) integers.} \]
\[ Z_p \text{ denotes the ring of } p\text{-adic integers.} \]
\[ Q_p \text{ denotes the } p\text{-adic number field.} \]
\[ K \text{ denotes } Q_p, \text{ the algebraic closure of } Q_p \text{ which contains } A \text{ the field of all algebraic numbers.} \]
\[ Z_p(G) \text{ denotes the group ring of } G \text{ with coefficients from } Z_p. \]
\[ \{C_i\} \text{ denotes the class sums of } G. \]
\[ \{K_i\} \text{ denotes the class sums of } H. \]
\[ \{e_i\} \text{ denotes the primitive central idempotents of } Q_p(G). \]
\[ h_i, k_i \text{ denotes the number of elements in } i\text{th conjugacy class of } G \text{ and } H \text{ respectively.} \]
\[ \{\chi_i\} \text{ denotes the absolutely irreducible characters of } Q_p(G). \]
\[ z_i \text{ denotes the degree of } \chi_i. \]

2. Theorem of Glauberman. We state our main theorem.

THEOREM 1. Let \( \theta: Z_p(G) \to Z_p(H) \) be an isomorphism. Then \( \theta(C_i) = \pm K_i \) for all \( i \).

Proof. Replacing \( \theta(G) \) by \( G \) we can assume that \( Z_p(G) = Z_p(H) \). We have to prove that \( C_i = \pm K_i \) for all \( i \). At first we claim that

\[ K_i = \sum_j a_{ij} C_j \quad \text{with } a_{ij} \in Z. \]

We know (see [1; p. 236]) that

\[ e_i = \frac{z_i}{(G : 1)} \sum \overline{\chi_i(g)} C_v, \quad g_v \in C_v, \]

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and that

\( C_t = \sum_i k_i \chi_{e_i}(g_i) \frac{e_i}{e_i} \), \quad g_i \in C_t. \)

By the same token we have

\( K_j = \sum_i k_j \chi_{i}(x_j) \frac{e_i}{e_i} \), \quad x_j \in K_j. \)

Substituting the value of \( e_i \) from (2) in (4) we obtain

\( K_j = \frac{1}{(G : 1)} \sum_{i,j} k_j \chi_{i}(x_j) \chi_{i}(g_i) C_t. \)

Also,

\( K_j = \sum_i a_{j,i} C_t \) for some \( a_{j,i} \in \mathbb{Z}. \)

Comparing (4) and (5) we have

\( a_{j,i} = \frac{1}{(G : 1)} \sum_{i,j} k_j \chi_{i}(x_j) \chi_{i}(g_i). \)

It follows from (7) that \( (G : 1) a_{j,i} \) is an algebraic integer. Since the \( p^n \)th cyclotomic polynomial over \( \mathbb{Q}_p \) is irreducible (see [2, p. 212]), by taking trace \( \mathbb{Q}_p(\xi)/\mathbb{Q}_p \) where \( \xi \) is an appropriate root of unity, we get from (7) that \( (G : 1) a_{j,i} \) is a rational number and hence a rational integer. But since \( a_{j,i} \) is a \( p \)-adic integer and \( (G : 1) \) is a \( p \)-power it follows that \( a_{j,i} \) is a rational integer. Hence (1) is established. Now we use the argument of Glauberman to conclude that \( a_{j,i} = \pm \delta_{j,i} \). This argument consists mainly of assigning a weight

\( w(K_1, \ldots, K_m) = \sum_{i,j} \chi_i(K_j) \chi_j(K_i) \)

to class sums of every group basis \( H \) and observing that

\( w(K_1, \ldots, K_m) = (G : 1) \sum_{i,j} k_j a_{i,j} \geq (G : 1)^2, \)

with equality if and only if for each \( i \) there is exactly one \( j \) such that \( a_{i,j} \neq 0 \) and for that \( j, a_{i,j} = \pm 1 \). Hence the class sums of any group basis \( H \) have weight \( (G : 1)^2 \) if and only if they are precisely \( \{ \pm C_i \} \). Reversing the role of \( G \) and \( H \) one obtains that the only class sums of a group basis with weight \( (G : 1)^2 \) are precisely \( \{ \pm K_i \} \). It follows therefore that \( \{ \pm C_i \} = \{ \pm K_i \} \).

3. Applications. We state two applications and indicate the proofs briefly as they are well known in the integral case and the proofs in this case are identical.

**Theorem 2.** Let \( \theta \) be an automorphism of \( \mathbb{Z}_p(G) \), where \( G \) is nilpotent of class 2. Then there exists an automorphism \( \lambda \) of \( G \) and a unit \( \gamma \) of \( \mathbb{Q}_p(G) \) such that

\( \theta(g) = \pm \gamma g \gamma^{-1} \) for all \( g \in G. \)
Proof. As in [5], the Theorem follows from Propositions 1 and 2.

Proposition 1. Let \( \theta \) be an automorphism of \( I(G) \) where \( I \) is an integral domain with field of quotients \( F \). Suppose that \( \theta(C_i) = C_i' \), and that there exists an automorphism \( \sigma \) of \( G \) such that \( \sigma(C_i) = C_i' \), for all \( i \). Then we can find a unit \( \gamma \in F(G) \) such that
\[
\theta(g) = \gamma g \gamma^{-1} \quad \text{for all} \quad g \in G.
\]

Proof. Proposition 1 has been proved for \( Z(G) \) in [5] but the proof is the same for any \( I(G) \). For Proposition 2, the existence of such a \( \sigma \) is proved in [6]. That \( \sigma(C_i) = C_i' \), follows just as in [5].

Passman and Whitcomb [3; 7] proved the next Theorem for \( Z(G) \).

Theorem 3. Let \( \theta: Z_p(G) \to Z_p(H) \) be an isomorphism. Then there exists a \( 1 - 1 \) correspondence \( N \to \phi(N) \) between normal subgroups of \( G \) and \( H \). This correspondence satisfies
\[
(1) \quad N_1 \subset N_2 \iff \phi(N_1) \subset \phi(N_2)
\]
\[
(2) \quad (N : 1) = (\phi(N) : 1)
\]
\[
(3) \quad (N_1, N_2) = (\phi(N_1), \phi(N_2)).
\]

Proof. The correspondence is established due to Theorem 1. The proofs of (1) and (2) are trivial, and (3) follows as in [4].

References


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