On a Polynomial Identity for $n \times n$ Matrices*

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Communicated by Claudio Procesi

Received December 1987

We prove that the polynomial

$$h_k(x_1, \ldots, x_k, y_1, \ldots, y_k) = \sum_{\sigma, \tau \in S_k} (\text{sgn} \tau) x_{\sigma(1)} y_{\tau(1)} \cdots x_{\sigma(k)} y_{\tau(k)}$$

vanishes on $n \times n$ matrices over a commutative ring for $k = 2n$ and for no smaller value of $k$. © 1989 Academic Press, Inc.

Let $C$ be a commutative ring with 1 and $M_n(C)$ the ring of $n \times n$ matrices over $C$. If $\{x_1, \ldots, x_k, \ldots\}$ and $\{y_1, \ldots, y_k, \ldots\}$ are two distinct sets of non-commuting variables for each $k \geq 1$ we define the polynomial

$$h_k(x_1, \ldots, x_k, y_1, \ldots, y_k) = \sum_{\sigma, \tau \in S_k} (\text{sgn} \tau) x_{\sigma(1)} y_{\tau(1)} \cdots x_{\sigma(k)} y_{\tau(k)},$$

where $S_k$ is the symmetric group of degree $k$.

It is clear that, for some $k$, $h_k$ is a polynomial identity for $M_k(C)$; in fact, since $h_k$ is alternating in the $x_i$ (and also in the $y_i$), $h_{n^2}$ is an identity for $M_n(C)$.

The purpose of this note is to prove that $2n$ is the smallest value of $k$ for which $h_k$ is a polynomial identity for $M_n(C)$. This answers a question of Formanek.

*: This research is supported by NSERC of Canada and MPI of Italy.
We have the following:

**Theorem.** \( h_{2n} \) is a polynomial identity for \( M_n(C) \). Moreover if \( h_k \) is a polynomial identity for \( M_n(C) \), then \( k \geq 2n \).

Our approach will be based on a proof of the Amitsur–Levitzki theorem given by Rosset in [1].

Before proceeding to the proof of this theorem we need some preliminaries.

Let \( E \) be the exterior algebra on a \( 4n \)-dimensional vector space \( V \) over the field of rational numbers \( \mathbb{Q} \) and let \( \{ v_1, ..., v_{4n} \} \) be a basis of \( V \) over \( \mathbb{Q} \). \( E \) may be viewed as the free algebra on \( V \) modulo the relations \( v_i v_j = -v_j v_i \). We write \( E = E_0 + E_1 \), where \( E_0 \) is the subalgebra generated by \( \mathbb{Q} \) and the monomials in the \( v_i \) of even degree and \( E_1 \) is the space generated by the monomials of odd degree.

In the following proposition we study some properties of the algebra \( M_k(E) \simeq M_k(\mathbb{Q}) \otimes_{\mathbb{Q}} E \).

**Proposition.** (i) If \( U \in M_k(E_0) \) is such that \( \text{tr}(U) = \text{tr}(U^2) = ... = \text{tr}(U^k) = 0 \), then \( U^k = 0 \).

(ii) If \( U, T \in M_k(E_1) \), then \( \text{tr}(UT) = -\text{tr}(TU) \).

**Proof.** Since \( E_0 \) is a commutative algebra over \( \mathbb{Q} \), (i) follows from Newton's formulas for symmetric functions (see [1]). To prove (ii), write \( U = \sum A_i w_i, \ T = \sum B_i w_i \), where \( A_i, B_i \in M_k(\mathbb{Q}) \) and the \( w_i \) are monomials in \( E_1 \). Recalling that \( \text{tr} \) is a symmetric bilinear form on \( M_k(\mathbb{Q}) \) and \( w_i w_j = -w_j w_i \), we have

\[
\text{tr}(AB) = \sum \text{tr}(A_i B_j) w_i w_j = - \sum \text{tr}(B_j A_i) w_j w_i = - \text{tr}(BA).
\]

**Proof of the Theorem.** Since, for \( k > 1 \),

\[
h_k(x_1, ..., x_k, y_1, ..., y_k) = \sum_{i,j=1}^k (-1)^{i+j} x_i y_j h_{k-1}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_k, y_1, ..., y_{j-1}, y_{j+1}, ..., y_k),
\]

to prove the second part of the theorem it is enough to check that \( h_{2n-1} \) is not an identity for \( M_n(C) \). To this end, consider the substitution (double staircase)
Then

\[ h_{2n-1}(e_{11}, ..., e_{21}, e_{12}, ..., e_{11}) = \sum e_{ii} \neq 0 \]

and \( h_{2n-1} \) is not an identity for \( M_n(C) \).

For the first part of the proof notice that, since \( h_{2n} \) is multilinear and each monomial has coefficient \( \pm 1 \), it is enough to prove that \( h_{2n} \) vanishes on \( M_n(Q) \) (see [2]).

Let \( A_1, ..., A_{2n}, B_1, ..., B_{2n} \in M_n(Q) \) and let

\[ A = \sum_{i=1}^{2n} A_i v_i, \quad B = \sum_{i=1}^{2n} B_i v_{2n+i}. \]

Then \( A, B \in M_n(E_1) \) and

\[ (AB)^{2n} = h_{2n}(A_1, ..., A_{2n}, B_1, ..., B_{2n}) v_1 v_{2n+1} v_2 v_{2n+2} \cdots v_{2n} v_{4n}. \]

This last equality can be verified by noticing that \( v_k^2 = 0 \) \( (k = 1, ..., 4n) \) and for \( \sigma, \tau \in S_{2n}, \)

\[ v_{\sigma(1)} v_{2n+\tau(1)} v_{\sigma(2)} v_{2n+\tau(2)} \cdots v_{\sigma(2n)} v_{2n+\tau(2n)} = (\text{sgn} \, \sigma \tau) v_1 v_{2n+1} v_2 v_{2n+2} \cdots v_{2n} v_{4n}. \]

Take now the matrix

\[ D = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \in M_{2n}(E_0). \]

Since for \( i \geq 1, (BA)^i B, A \in M_n(E_1) \), by Proposition (ii), \( \text{tr}((AB)^i) = -\text{tr}((BA)^i) \); thus

\[ \text{tr}(D') = \text{tr}((AB)^i) + \text{tr}((BA)^i) = 0. \]

But then Proposition (i) forces \( D^{2n} = 0 \) and so \( (AB)^{2n} = 0 \). This last equality is equivalent to \( h_{2n}(A_1, ..., A_{2n}, B_1, ..., B_{2n}) = 0 \). The proof is now complete.

It has come to our attention that a different proof of the above theorem has been announced by Qing Chang.

REFERENCES


Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium