Asymptotic Distribution of Some Multivariate "Success Run" Renewal Processes, Applied to a 2-i.i.d. Unit Repairable System

Doug Wiens*

Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 4H8

If the probability of "failure" in a multivariate renewal process of the "success run" type is very small, then if certain conditions are imposed on the components of the renewals, the joint distribution of their total durations is approximately exponential with all mass along one line. This result is applied to a 2-i.i.d. unit repairable system of the "1 out of 2:G, Cold Standby" type.

1. INTRODUCTION

A k-dimensional success run renewal process with constant probability 1 - p of "success" may be represented as a sum $Z = \sum_{m=1}^{v} X_m + Y$ of independent random vectors. Here $X_m$ is the vector of times spent by various components of the process during the $m$th success, $Y$ is the vector of such times during the failure, and $Z$ is the vector of such times until the termination of the first failure. The number of terms, $v$, is a geometrically distributed random variable with parameter $p$. Suppose that $E[v] = p^{-1}$ is very large. Then, under certain conditions made explicit in Lemma 1, $Z$ has an approximate exponential distribution with all of its mass along one line in $k$ space.

The result of the lemma is applied to a 2-i.i.d. unit repairable system of the "1 out of 2:G, Cold Standby" type. In the model considered, only one unit operates at a time. When it fails, both units enter a switching period of random length. The failed unit is switched into the repair phase, and the idle unit is switched into the operating phase. A newly repaired unit is assumed to be as good as new. The system is up only when a unit is functioning, so that it is down during each switching period and comes down again when a unit fails before the other is repaired. Parameter $p$ described above is the probability of this latter event. After this second type of downtime, that is, when the repair is finally completed, the newly repaired unit is switched on, the failed unit is switched into repair, and a new i.i.d. "cycle" commences.

A "success" is a repair before a unit failure, followed by a switching period. A "failure" is a unit failure before a repair, followed by a switching period. The components of (i) $X_m$, (ii) $Y$, and (iii) $Z$ are taken to be the total uptime, repair time, switching time, and idle time (a unit neither operating, nor being repaired, nor being switched) during (i) the $m$th success, (ii) the failure, or (iii) the cycle.

For this system, the exact Laplace transform of the distribution of $Z$ is readily

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obtainable (see [1] and [2] and equation (3) below). The determination of the distribution itself, in closed form, is in general a very intractable problem, however. The approximation given here is of a particularly simple form. Conditions under which it is valid are determined and evaluated for some classes of distributions. For these the approximation is valid provided only that \( E[v] \) is sufficiently large.

2. DERIVATIONS

Notation

Let \( \{v_n, X_{n,m}, Y_n; n, m = 1, 2, \ldots \} \) be independent random variables and \( k \)-dimensional nonnegative random vectors, where

(a) \( P(v_n = i) = q_n^{-1}p_n (0 \leq p_n \leq 1, q_n = 1 - p_n, i = 1, 2, \ldots) \);
(b) \( E[Y_n] = \beta_n, E[Y_n Y_n'] = B_n \); and
(c) the \( X_{n,m} \) are identically distributed for fixed \( n \), with

\[ E[X_{n,m}] = \alpha_n, E[X_{n,n} X_{n,m}'] = A_n. \]

Let \( Z_n = \sum_{m=1}^{n-1} X_{n,m} + Y_n \), and define

\[ \gamma_n = E[Z_n], C_n = E[Z_n Z_n'], \]
\[ \Gamma_n = \text{diag}(\gamma_n^{-1}, \ldots, \gamma_n^{-1}), \]
\[ \hat{Z}_n = \Gamma_n Z_n = \left( \frac{Z_n1}{E[Z_n1]}, \ldots, \frac{Z_nk}{E[Z_nk]} \right). \]

**Lemma 1:**

\[ \gamma_n = \beta_n + \frac{q_n}{p_n} \alpha_n, \quad (1) \]

and

\[ C_n = \left( B_n + \frac{q_n}{p_n} A_n \right) + (\gamma_n - \beta_n)\gamma_n' + \gamma_n(\gamma_n - \beta_n)'. \quad (2) \]

If (i) \( \lim \text{tr} \Gamma_n \left( B_n + \frac{q_n}{p_n} A_n \right) \Gamma_n = 0 \),

then \( \hat{Z} = \lim_{n \to \infty} \hat{Z}_n \) is distributed as \( k \) copies of the same exponential random variable with mean 1; that is,

\[ P(\hat{Z}_i > t; i = 1, 2, \ldots, k) = \exp\{-\max_{1 \leq i \leq k} t_i\}. \]

**Remark:** Condition (i) implies \( E[v_n] \to \infty \), since

\[ \text{tr} \Gamma_n \left( B_n + \frac{q_n}{p_n} A_n \right) \Gamma_n \geq k p_n^2, \]

and is equivalent to

(ii) \( E[(\hat{Z}_{n,i} - \hat{Z}_{n,j})^2] \to 0, \quad \frac{E[Y_n^2]}{\gamma_n^2} \to 0, \quad \text{var}(\hat{Z}_{n,i}) \to 1, \)

for \( i, j = 1, 2, \ldots, k \).
PROOF: Let \( s = (s_1, \ldots, s_k)' \) be a vector of parameters for Laplace–Stieltjes transforms, put \( 1 = (1,1, \ldots, 1)' \), let \( O(A_n) \) be a matrix each of whose elements is “big oh” of the corresponding element of \( A_n \) as \( n \to \infty \), and define \( O(B_n) \) similarly.

Conditioning on \( v_n \),

\[
E[e^{-s'v_n}] = \frac{p_n E[e^{-s'v_n}]}{1 - q_n E[e^{-s'v_n}]},
\]

from which Eqs. (1) and (2) are obtained. By Eq. (1) and Taylor's theorem, Eq. (3) is

\[
p_n[1 - s'\beta_n + s'O(B_n)s] \quad \frac{1 - s'\beta_n + s'O(B_n)s}{1 - q_n[1 - s'\alpha_n + s'O(A_n)s]} = \frac{1}{1 + s'(\gamma_n - \beta_n) - (q_n/p_n)s'O(A_n)s}.
\]

Then

\[
E[e^{-s'v_n}] = \frac{1}{1 + s'[1 - s'T_n\beta_n - (q_n/p_n)s'T_nO(A_n)\Gamma_n]} \quad \to (1 + s'[1])^{-1} \text{ as } n \to \infty,
\]

by (i) and the Cauchy–Schwartz inequality. Now let \( U \) be a random variable with \( P(U > t) = e^{-t} \), put \( U_1 = \cdots = U_k = U \), and let \( U = (U_1, \ldots, U_k)' \). Then \( E[e^{-s'u}] = (1 + s'[1])^{-1} \), so that \( \tilde{Z} \) is distributed as \( \tilde{Z} \) with

\[
P(\tilde{Z}_i > t_i; i = 1, \ldots, k) = P(U > t_i; i = 1, \ldots, k) = \exp(-\max_{1 \leq i \leq k} t_i). \quad \text{Q.E.D.}
\]

3. AN APPLICATION

For the repairable system described in the Introduction, define, for \( n = 1,2, \ldots \),

- \( T_n = \) operating time of a unit, with cdf \( F_n(t) \), \( \bar{F}_n = 1 - F_n \);
- \( R_n = \) repair time for a unit, with cdf \( G_n(t) \), \( \bar{G}_n = 1 - G_n \);
- \( S_n = \) switching time for the units, with cdf \( H_n(t) \), \( E[e^{-s_n}] = \hat{H}_n(s) \).

The possible changes of state in the two types of renewals are illustrated below. In each figure a solid line indicates a unit operating period, a broken line a unit repair period, a box a switching period. A “success” is illustrated on the left, a “failure” on the right:

To apply the lemma, set \( k = 4 \) and define

\[ Z_{n,1} = \text{total uptime in a cycle}, \]
\[ Z_{n,2} = \text{total repair time in a cycle}, \]
\[ Z_{n,3} = \text{total switching time in a cycle}, \]
\[ Z_{n,4} = \text{total idle time in a cycle}. \]
Define the components of $X_n$ and $Y_n$ correspondingly, but restricted to successes and failures, respectively. In the two figures above

$$X_n = (y, x, z - y, x - y),$$

$$Y_n = (y, x, z - x, x - y).$$

Then

$$p_n = P(R_n > T_n) = E[\tilde{G}_n(T_n)],$$

$$E[e^{-y'X_n}] = \frac{\hat{H}_n(s_3)}{q_n} \int_{0}^{\infty} \int_{0}^{\infty} e^{-s_3 y - s_2 x - s_1 z - s_4} dG_n(x) dF_n(y),$$

and

$$E[e^{-y'Y_n}] = \frac{\hat{H}_n(s_3)}{p_n} \int_{0}^{\infty} \int_{0}^{\infty} e^{-s_3 y - s_2 x - s_1 z - s_4} dG_n(x) dF_n(y),$$

from which $E[e^{-y'Z_n}]$ is obtainable, as at Eq. (3). One then finds

$$\gamma_{n,1} = p_n^{-1} E[T_n], \quad \gamma_{n,2} = p_n^{-1} E[R_n], \quad \gamma_{n,3} = p_n^{-1} E[S_n],$$

$$\gamma_{n,4} = p_n^{-1} \left\{ E[T_n] - E[R_n] + 2 \int_{0}^{\infty} F_n(t) \tilde{G}_n(t) dt \right\}.$$  \hspace{1cm} (6)

The diagonal elements of $\Gamma_n(B_n + (q_n/p_n)A_n)\Gamma_n$ are

$$\frac{p_n E[T_n^2]}{E^2[T_n]}, \quad \frac{p_n E[R_n^2]}{E^2[R_n]}, \quad \frac{p_n E[S_n^2]}{E^2[S_n]},$$

and

$$\frac{p_n E[(T_n - R_n)^2]}{E^2[T_n - R_n] + 2 \int_{0}^{\infty} F_n(t) \tilde{G}_n(t) dt} \leq \frac{p_n E[(T_n - R_n)^2]}{E^2[T_n - R_n]}.$$  \hspace{1cm} (7)

The lemma then implies

**COROLLARY 1:** Let $Z_n = (Z_{n,1}, \ldots, Z_{n,4})'$, $E[Z_n]$ and $p_n$ be as exhibited at Eqs. (4)–(6). If, as $n \to \infty$, the following condition is satisfied:

(iii) $p_n E[W_n^2]/E^2[W_n] \to 0$ for $W = T, R, S$ and $T - R$,

then the distribution of $Z_n$ is asymptotic to

$$P(Z_{n,i} > t_i; i = 1, 2, 3, 4) = \exp \left( -\max_{1 \leq i \leq 4} \frac{t_i}{\gamma_{n,i}} \right).$$  \hspace{1cm} (7)

The marginal means are exact for all $n$.

**REMARKS:** (1) Condition (iii) is equivalent to

(iv) $E[v_n] \to \infty$;

(v) the coefficients of variation of $T_n, R_n, S_n, T_n - R_n$ are all $o(E[v_n])^{1/2}$.
(2) Of interest also are the distributions of

\[ Z_{n,5} = \text{total cycle time} \]
\[ Z_{n,6} = \text{total downtime in a cycle} \]

but these are derivable from

\[ Z_{n,5} = (Z_{n,1} + Z_{n,2} + Z_{n,4} + 2Z_{n,3})/2 \quad Z_{n,6} = Z_{n,5} - Z_{n,1}. \]

**Special Cases**

1. Assume that \( S_n \) is independent of \( n \). Let \( T_n \) have a log-normal distribution with mean \( \epsilon_n^{-1}e^{\mu_n/2} \) and coefficient of variation \( (e^{\mu_n} - 1)^{1/2} \). Let \( R_n \) have a lognormal distribution with mean \( \delta_n^{-1}e^{\nu_n/2} \) and coefficient of variation \( (e^{\nu_n} - 1)^{1/2} \). Then

\[
p_n = P\left( \log R_n - \log T_n > 0 \right)
= P\left( \Phi > \frac{\log(\delta_n/\epsilon_n)}{\sqrt{\mu_n + \lambda_n}} \right) \quad (\Phi \text{ a standard normal variate})
\rightarrow 0
\]

as \( \delta_n/\epsilon_n \rightarrow \infty \) while \( \mu_n \) and \( \lambda_n \) remain bounded, or as \( \mu_n, \lambda_n \rightarrow 0 \) while the ratio \( \delta_n/\epsilon_n \) remains bounded below by \( 1 + \alpha \) for some \( \alpha > 0 \). Either of these conditions implies (v) above, so that in this case the exponential approximation (7) is valid provided \( E[v_n] \) is sufficiently large for either of the above reasons.

2. Assume that \( S_n \) is independent of \( n \), while \( T_n \) is Erlang in \( a_n \) stages with mean \( a_n/\lambda_n \) and \( R_n \) is Erlang in \( b_n \) stages with mean \( b_n/\mu_n \). Then

\[
p_n = P\left( \frac{R_n}{T_n} > 1 \right) = P\left( F_{1/2}^{1/(2a_n^2)} > \frac{E[T_n]}{E[R_n]} \right) \rightarrow 0
\]

if \( E[T_n]/E[R_n] \rightarrow \infty \) for fixed \( a_n \) and \( b_n \), or if the variances of \( T_n \) and \( R_n \) both tend to zero while the ratio \( E[T_n]/E[R_n] \) remains bounded below by \( 1 + \alpha \) for some \( \alpha > 0 \).

**REFERENCES**
