STAT 312 Lab 12

1. Suppose that $X_1, ..., X_n$ are i.i.d. r.v.s with density $p(x; \lambda) = \lambda e^{-\lambda x} / (1 - e^{-\lambda})$ for $0 \leq x \leq 1$. This is the density of an $E(\lambda)$ (exponential with parameter $\lambda$) r.v., subject to the restriction that it lie in $[0, 1]$:

$$p(x; \lambda) = \frac{d}{dx} P(X \leq x | 0 \leq X \leq 1) = \frac{d}{dx} \frac{P(X \leq x \text{ and } 0 \leq X \leq 1)}{P(0 \leq X \leq 1)} = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda}}.$$

(a) Derive the Newton-Raphson iteration scheme for the computation of the maximum likelihood estimator $\hat{\lambda}$. Suggest a starting value. (As regards starting values there is no ‘right’ answer - just use your intuition to suggest something reasonable.)

(b) Describe the asymptotic distribution of $\hat{\lambda}$, including an explicit expression for the variance.

2. Discuss the computation of the least squares estimates $\left(\hat{\alpha}, \hat{\beta}\right)$, obtained from the Gauss-Newton method, in the nonlinear regression model with exponential response

$$Y_i = \alpha e^{-\beta x_i} + \varepsilon_i.$$

Write down the scheme explicitly. Suggest a method of obtaining starting values, based on the observation that, if the random error is ignored, then $\log Y$ is a linear function of the parameters.

3. Suggest a method of obtaining approximate 95% confidence intervals on each of the two parameters in the previous question.

4. Complete the example of Lecture 32, in order to obtain the ‘Student’s’ $t_{n-1}$ density

$$g_{n-1}(t) = \frac{\Gamma \left(\frac{n}{2}\right)}{\sqrt{n} \Gamma \left(\frac{n-1}{2}\right)} \left(1 + \frac{t^2}{n-1}\right)^{-\frac{n}{2}}, -\infty < t < \infty.$$

[Hint: You can differentiate $G_{n-1}(t)$, obtained in class, under the integral sign. Then make a change of variables in the resulting integral so as to recognize it as one for which you know the value.]

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5. Suppose that $X \sim N(\mu, \sigma^2)$. A ‘normalized mean’ is

$$NM = \frac{\mu}{\sigma},$$

and expresses the mean as a multiple of the standard deviation (and hence has no units associated with it). Find an unbiased estimator of $NM$, based on a sample $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. [Hint: Start by looking at the mle, or at $\bar{X}/S$; evaluate its expected value by exploiting what you have learned about the distribution of $(\bar{X}, S)$ in Normal samples.]

6. Let $X_1, \ldots, X_n$ be a sample from a $N(\mu, \sigma^2)$ population, and let $\hat{cv}$ be the mle of the coefficient of variation $cv = \sigma/\mu$. Thus the parameter vector is $\theta = (\mu, \sigma^2)'$ and $cv$ is a function $\tau(\theta)$. Obtain the approximate normal distribution of $\hat{cv}$, including an explicit expression for the variance (which should turn out to be $cv^2 \left( \frac{1}{2} + cv^2 \right)/n$).