STAT 312 Lab 2

1. Let

\[ X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 3 & 2 \end{pmatrix}, \]

as in Lab 1 Problem 1. Exhibit an orthonormal basis for \( \text{col} (X) \), with appropriate verifications.

2. Let \( \mathbf{x} \) be a random vector with mean vector \( \mu \) and covariance matrix \( \Sigma \). Show that

\[ E [\mathbf{x}' A \mathbf{x}] = tr (A \Sigma) + \mu' A \mu. \]

Note: I will take a dim view of a solution which starts by expanding \( \mathbf{x}' A \mathbf{x} \) as a double sum, and taking the sum of expectations. I want you to use properties of the trace, and of expectations, starting with the observation that the trace of a number - such as \( \mathbf{x}' A \mathbf{x} \) - is the number itself, then continuing by applying other properties of the trace.

For instance if \( M \) is a square random matrix then

\[ E [tr \{M\}] = E \left[ \sum_i M_{ii} \right] = \sum_i E [M_{ii}] = tr \{E[M]\}. \]

3. (a) What do we mean when we say that vectors \( \mathbf{u}_1, \ldots, \mathbf{u}_n \) are ‘linearly independent’?

(b) Show that if \( \mathbf{u}_1, \ldots, \mathbf{u}_n \) are mutually orthogonal non-zero vectors, then they are linearly independent.

4. (a) Define what we mean by the ‘transition matrix for a Markov chain with \( s \) states’.

(b) Suppose that \( P \) is the transition matrix for a Markov chain with \( s \) states. Show that \( \mathbf{1}_s \) is an eigenvector of \( P \). What is the eigenvalue?
5. Consider a regression model in which one makes \(n\) observations on a variable \(Y\), which varies with regressors \(X_1, \ldots, X_{p-1}\) according to
\[
Y = \beta_0 + X_1\beta_1 + \cdots + X_{p-1}\beta_{p-1} + \varepsilon.
\]
In matrix terms,
\[
y = X\beta + \varepsilon,
\]
where \(X\) is the \(n \times p\) matrix with columns \(1_n, z_1, \ldots, z_{p-1}\) and \(z_i\) contains all \(n\) values of the variable \(X_i\). Assume that the rank of \(X\) is \(p\), so that the ‘hat’ matrix is \(H = X(X'X)^{-1}X'\).

(a) Show that the sum of the elements in any row of \(H\) is one. (\textit{Hint:} \(HX = X\); what is the first column?)

(b) Show that the average of these diagonal elements is \(\bar{h} = p/n\).

(c) Let \(\hat{\beta}\) be the vector of LSEs, \(\hat{y} = X\hat{\beta}\) the vector of ‘fitted values’, and \(e = y - \hat{y}\) the vector of residuals. Assuming that the errors \(\varepsilon_i\) are i.i.d., with mean 0 and variance \(\sigma^2\), show that the covariance matrices of \(\hat{y}\) and \(e\) are \(\sigma^2H\) and \(\sigma^2(I - H)\) respectively. [Note: This result implies that the variance of the \(i^{th}\) residual is \(\text{VAR}[e_i] = \sigma^2(1 - h_{ii})\), so that if \(h_{ii}\) is near 1, the corresponding residual must be near its expected value of 0 and \(\hat{y}_i\) must be near \(x_i'\hat{\beta}\), regardless of the observed value \(y_i\). When this happens \(x_i\) is called a “highly influential” value, and the \(h_{ii}\) are called ‘influence measures’ - they are important tools in regression diagnostics.]

(d) Show that the \(i^{th}\) diagonal element of \(H\) is \(h_{ii} = x_i'(X'X)^{-1}x_i\), where \(x_i'\) is the \(i^{th}\) row of \(X\), and that \(0 \leq h_{ii} \leq 1\).

6. (a) Define what we mean when we say that a matrix is symmetric and idempotent.

(b) Let \(A_{n \times n}\) and \(B_{n \times n}\) be symmetric idempotent matrices. Show that \(A - B\) is idempotent if \(AB = B\).