Outline of theory you are expected to know for Exam III.

- Any zero-mean, weakly stationary time series may be approximated arbitrarily closely by a series \( \{X_{t,N}\}_{t=-\infty}^{\infty} \) of the form

\[
X_{t,N} = \sum_{k=0}^{N} \{A_k \cos(\lambda_k t) + B_k \sin(\lambda_k t)\}
\]

for Fourier frequencies \( \lambda_k \in [-\pi, \pi] \). Here \( A_0, \ldots, A_N, B_0, \ldots, B_N \) are uncorrelated, zero-mean r.v.s with \( \text{VAR}[A_k] = \text{VAR}[B_k] = \sigma_k^2 \).

- The ACF of \( \{X_{t,N}\} \) was derived and expressed as

\[
\rho(m) = E[\cos(\Lambda m)]
\]

for some discrete, symmetric r.v. \( \Lambda \) with possible values \( \lambda_0 = 0, \pm \lambda_1, \ldots, \pm \lambda_N \).

- As we improve the approximation by letting \( N \to \infty \), the distribution of \( \Lambda \) tends to one with a density. Now we write \( \lambda = 2\pi \nu, -1/2 \leq \nu \leq 1/2, \)
and write the density of $\nu$ in the form $f(\nu)/\sigma^2$. The relationship above becomes (how?)

$$\frac{\gamma(m)}{\sigma^2} = \rho(m) = \int_{-1/2}^{1/2} \cos(2\pi \nu m) \frac{f(\nu)}{\sigma^2} d\nu$$

$$= \frac{1}{\sigma^2} \int_{-1/2}^{1/2} e^{2\pi i \nu m} f(\nu) d\nu.$$ 

• These transforms are all invertible:

$$\gamma(m) = \int_{-1/2}^{1/2} e^{2\pi i \nu m} f(\nu) d\nu \iff \sum_{m=-\infty}^{\infty} e^{-2\pi i \nu m} \gamma(m).$$

A consequence is (how?)

$$f(\nu) = \gamma(0) + 2 \sum_{m=1}^{\infty} \{\cos(2\pi \nu m) \cdot \gamma(m)\}.$$
• Similarly the CCF and cross-spectrum are related by

$$\gamma_{XY}(m) = \int_{-1/2}^{1/2} e^{2\pi i \nu m} f_{XY}(\nu) d\nu,$$

$$f_{XY}(\nu) = \sum_{m=-\infty}^{\infty} e^{-2\pi i \nu m} \gamma_{XY}(m).$$

• The squared coherence is defined by

$$\rho^2_{Y,X}(\nu) = \frac{|f_{YX}(\nu)|^2}{f_Y(\nu) f_X(\nu)} \in [0, 1].$$

The 1 is attained (why? - you should be able to give the derivation) if

$$Y_t = \sum_{s=-\infty}^{\infty} a_s X_{t-s}$$

for constants \( \{a_s\}_{s=-\infty}^{\infty} \). In this latter case we say \( \{Y_t\} \) is a linear filter of \( \{X_t\} \).
If \( \{Y_t\} \) is a filter of \( \{X_t\} \) then

\[
f_Y(\nu) = |A(\nu)|^2 f_X(\nu)
\]

and

\[
f_{YX}(\nu) = f_X(\nu)A(\nu)
\]

(recall you were asked to derive this latter equality) where

\[
A(\nu) = \sum_{s=-\infty}^{\infty} a_s e^{-2\pi i \nu s}
\]

is the IFT. One can invert this to get the \( a_s \)'s from \( A(\nu) \) - how?

One consequence - If \( \{X_t\} \) is MA(q) it can be viewed as a filter of \( \{w_t\} \), leading to the result

\[
f_X(\nu) = |\theta(e^{-2\pi i \nu})|^2 \sigma_w^2
\]

Similarly if \( \{X_t\} \) is AR(p) then \( \{w_t\} \) can be viewed as a filter of \( \{X_t\} \), leading to an expression for the power of \( \{X_t\} \) - what is it? Derive the power of an ARMA(1,1) series.
• Estimating the power. Since $f(\nu)$ is the IFT of the ACF, a natural estimate is the corresponding IFT of the estimated ACF:

$$\hat{f}(\nu_k) = \sum_{m=-(n-1)}^{n-1} e^{-2\pi i \nu_k m} \hat{\gamma}(m) = |X(k)|^2,$$

(the “periodogram”) where

$$X(k) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{-2\pi i \nu_k t} x_t$$

is the DFT of the data. The data can be recovered from this:

$$x_t = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{2\pi i \nu_k t} X(k);$$

you should know the derivation of this result.

• For various reasons (what are they?) one generally computes a smoothed version of the periodogram:

$$\hat{f}(\nu_k) = \frac{1}{L} \sum_{l=-L^{-1}}^{L-1} |X(k + l)|^2.$$
The approximate distribution is

$$\hat{f}(\nu_k) \overset{d}{\approx} \frac{f(\nu_k)}{df} \chi^2_{df}$$

with \(df = 2Ln/n'\). What then are the approximate mean and variance? Derive the form of a confidence interval on \(f(\nu_k)\).

- One application is “lagged regression” or “impulse-response” problems: If an examination of the estimated coherence indicates that series \(\{X_t\}, \{Y_t\}\) are strongly coherent at some frequencies, then we might try to fit a model of the form

$$Y_t = \sum_{s=-\infty}^{\infty} \beta_s X_{t-s} + \nu_t.$$  

You should be able to write down the MSE and differentiate it w.r.t. each \(\beta_r\) so as to obtain the equations

$$\gamma_{YX}(r) = \sum_{s=-\infty}^{\infty} \beta_s \gamma_X(r-s), \quad r = 0, \pm 1, \pm 2, \ldots.$$
From this, derive the equation

\[ B(\nu) = \frac{f_{YX}(\nu)}{f_X(\nu)}, \]

where \( B(\nu) \) is the IFT of \( \{\beta_s\} \). How then is \( \hat{\beta}_s \) obtained?

- Another application is the construction of filters. If an examination of the estimated spectrum of a series reveals certain ranges of interesting frequencies, we might choose a frequency response function \( A(\nu) \) to be large at these frequencies, small otherwise. In theory, the original and filtered spectra are related through

\[ f_Y(\nu) = |A(\nu)|^2 f_X(\nu). \]

On R we take the desired \( A(\nu) \), and then do an approximate integration (how?) so as to obtain \( M \) filter coefficients \( a_s^M \) approximating those of \( A(\nu) \). Then the filtered series can be computed - what will it be? - and has spectrum

\[ |A_M^M(\nu)|^2 f_X(\nu), \]

where \( A_M^M(\nu) = (\text{what?}) \).