Derivation of the density of a noncentral chi-square random variable

Preliminaries:

• First write, as in Assignment 1 question 4(b),
  \[ X^2 = X_1^2 + X_{m-1}^2, \]
  where \( X_1^2 \sim \chi_1^2 (\lambda^2) \), independently of \( X_{m-1}^2 \sim \chi_{m-1}^2 \).

• Recall that if \( K \) is a r.v. with a Poisson distribution and mean \( \lambda^2/2 \), then the distribution is
  \[ P_\lambda (K = k) = e^{-\lambda^2/2} (\lambda^2/2)^k / k!, \quad k = 0, 1, 2, ... . \]

• The (central) \( \chi_{2k+1}^2 \) density is
  \[ f_{2k+1} (x) = e^{-x/2} \frac{(x/2)^{k-\frac{1}{2}}}{2\Gamma(k + \frac{1}{2})}. \]

• A useful and easily derived relationship between factorials and gammas is
  \[ \frac{(2k)!}{2^{2k} k!} = \frac{\Gamma(k + \frac{1}{2})}{\sqrt{\pi}}. \] (1)

  Recall that \( \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \).

Result: The density of \( X_1^2 \) is

\[ \sum_{k=0}^{\infty} P_\lambda (K = k) f_{2k+1} (x). \] (2)

We write this as

\[ X^2 \sim \chi_{2K+1}^2; \]
the interpretation is that \( X_1^2 \) has the central \( \chi_{2K+1}^2 \) distribution, i.e. chi-square with random degrees of freedom \( 2K + 1 \), with \( K \) following a Poisson distribution with mean \( \lambda^2/2 \). Thus, adding on \( X_{m-1}^2 \), we have that

\[ X^2 \sim \chi_{2K+m}^2; \]
with density

\[ \sum_{k=0}^{\infty} P_\lambda (K = k) f_{2k+m} (x); \]
here \( f_{2k+m} \) is the \( \chi_{2k+m}^2 \) density. If \( \lambda = 0 \) then \( P (K = 0) = 1 \), so that \( X^2 \) has the central \( \chi_m^2 \) distribution.
Derivation of (2): Since $T = X_1 - \lambda \sim N(0,1)$, with distribution $\Phi(t)$ and density $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$, the distribution of $X_1^2$ is
\[
P (X_1^2 \leq x) = P (-\sqrt{x} \leq X_1 \leq \sqrt{x}) = P (-\sqrt{x} - \lambda \leq X_1 - \lambda \leq \sqrt{x} - \lambda) = \Phi(\sqrt{x} - \lambda) - \Phi(-\sqrt{x} - \lambda);
\]
hence the density is
\[
\frac{d}{dx} \left[ \Phi(\sqrt{x} - \lambda) - \Phi(-\sqrt{x} - \lambda) \right]
= \frac{1}{2\sqrt{x}} \left[ \phi(\sqrt{x} - \lambda) + \phi(\sqrt{x} + \lambda) \right]
= \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2}(x+\lambda^2)} \left[ \frac{e^{\lambda\sqrt{x}} + e^{-\lambda\sqrt{x}}}{2} \right]
= \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2}(x+\lambda^2)} \left[ \sum_{k=0}^{\infty} \frac{(\lambda\sqrt{x})^{2k}}{(2k)!} \right]
= \sum_{k=0}^{\infty} \left\{ e^{-\frac{\lambda^2}{2} \frac{(\lambda^2/2)^k}{k!}} \right\} \left\{ e^{-\frac{x}{2}} \frac{x^k}{k!} \frac{2^{2k}k!}{(2k)!} \frac{1}{\sqrt{2\pi x}} \right\}
\]
\[
= \sum_{k=0}^{\infty} P_\lambda (K = k) f_{2k+1} (x) \left\{ 2\Gamma \left( k + \frac{1}{2} \right) \left( \frac{x}{2} \right)^{1/2} \frac{2^{2k}k!}{(2k)!} \frac{1}{\sqrt{2\pi x}} \right\}
\]
\[
= \sum_{k=0}^{\infty} P_\lambda (K = k) f_{2k+1} (x) \left\{ \Gamma \left( k + \frac{1}{2} \right) \frac{2^{2k}k!}{(2k)!} \frac{1}{\sqrt{\pi}} \right\}
\]
\[
= \sum_{k=0}^{\infty} P_\lambda (K = k) f_{2k+1} (x),
\]
using (1).