1. Let \( \hat{\gamma}_{XY}(m) \) be the usual sample cross-covariance function for jointly stationary series \( \{X_t, Y_t\} \), with data \( \{x_t, y_t\}_{t=1}^{n} \). Show that
\[
E[\hat{\gamma}_{XY}(m)] = \left( \frac{n - m}{n} \right) \{\gamma_{XY}(m) - \text{COV}[\bar{y}, \bar{x}^*] - \text{COV}[\bar{x}, \bar{y}^*] + \text{COV}[\bar{x}, \bar{y}]\},
\]
where \( \bar{x}^* \) is the average of \( \{x_t\}_{t=m+1}^{n} \) and \( \bar{y}^* \) is the average of \( \{y_t\}_{t=1}^{n-m} \).

2. Assume \( \mu_X = 0 \); consider the problem of minimizing the function
\[
f_m(\alpha_{1,m}, ..., \alpha_{m,m}) = E \left[ \{X_t - \alpha_{1,m}X_{t-1} - ... - \alpha_{m,m}X_{t-m}\}^2 \right],
\]
which is the MSE when \( X_t \) is forecast by \( \alpha_{1,m}X_{t-1} + ... + \alpha_{m,m}X_{t-m} \). Let the minimizers be \( \alpha_{1,m}^*, ..., \alpha_{m,m}^* \). The lag-m PACF value, written \( \phi_{mm} \), is defined to be \( \alpha_{m,m}^* \). Show that
\[
\phi_{mm} = \text{corr} \left[ X_t - \hat{X}_t, X_{t-m} - \hat{X}_{t-m} \right],
\]
where each \( \hat{X} \) denotes the best (i.e. minimum MSE) predictor which is a linear function of \( X_{t-1}, ..., X_{t-m+1} \).

3. The obvious estimate of
\[
f(\nu_k) = \sum_{m=-\infty}^{\infty} e^{-2\pi i\nu_k m} \gamma(m),
\]
using data \( \{x_t\}_{t=1}^{n} \) is
\[
\hat{f}(\nu_k) = \sum_{m=-n+1}^{n-1} e^{-2\pi i\nu_k m} \hat{\gamma}(m).
\]
Show that this reduces to the periodogram:
\[
\hat{f}(\nu_k) = I(\nu_k) = |X(k)|^2.
\]

4. Suppose that \( \{X_t\} \) is AR(2), mean zero and stationary.
   (a) In the representation
   \[
   X_t = w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + ...
   \]
   show that
   \[
   \psi_k = \phi_1 \psi_{k-1} + \phi_2 \psi_{k-2}, \text{ where } \psi_1 = \phi_1, \psi_0 = 1.
   \]
(b) Obtain a closed form expression for $\psi_k$. (*Hint: write* $\psi_k - b\psi_{k-1} = a (\psi_{k-1} - b\psi_{k-2})$ for suitable constants $a$ and $b$; then iterate.)

5. Suppose that filter coefficients are approximated by

$$a_s^M = \frac{1}{M} \sum_{k=0}^{M-1} A(\omega_k) e^{2\pi i \omega_k s}$$

and then $Y_t^M = \sum_{|s|<M/2} a_s^M X_{t-s}$ for $t = M/2 - 1, ..., n-M/2$.

(a) Show that these coefficients are real and symmetric ($a_s^M = a_{-s}^M$), if $A$ is real and symmetric: $A(\omega) = A(-\omega)$.

(b) Show that

$$a_s^M = \sum_{t=-\infty}^{\infty} a_t \left[ I(t - s \text{ is a multiple of } M) \right].$$

(c) Establish the bound in Problem 4.32:

$$E \left[ (Y_t^M - Y_t)^2 \right] \leq 4 \gamma_X(0) \left( \sum_{|k| \geq M/2} |a_k| \right)^2.$$